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Packet Radio Temporary Note # 114
SRI Temporary Note #1

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19 August 1974

MINIMUM CROSS INTERFERENCE POWER IN ASYNCHRONOUS MULTIPLE ACCESS SYSTEMS

Introduction

In multiple-access communication systems it is desirable to choose signals so that the inter-user interference is minimized. When the users are synchronized, this can be done by choosing a set of signaling waveforms which are orthogonal over the signaling interval. Since there are many such sets, additional implementation restrictions such as peak-power constraints can be applied to select a signal set. When these constraints are applied it is often found that some form of spread-spectrum signal is most suitable.

When the users lack synchronization, the signal design problem is more difficult. One solution is to use frequency-division techniques, since signal sets which are almost orthogonal are easily generated. Occasionally, frequency-division is ruled out, so that all signals must have spectra which occupy the same channel. The purpose of this note is to examine this situation and develop an expression for the minimum cross-interference power, to find a lower bound for this expression and to find signal sets which attain the minimum.

Cross-Interference Power

More specifically, suppose that the communication receiver uses a matched-filter detector; i.e., it is optimum for an additive white Gaussian noise channel. Then the detector output will be the cross-correlation function of the signal and the receiver input. To a first approximation, the detector performance will be a monotonic function of the received signal-to-noise ratio; hence, performance in a multi-user environment will be optimized by minimizing the interfering power.

Let $s_i(t)$, $|i = 1, 2, \cdots M|$ represent a set of signals, each of duration T, and let $S_i(f)$ be the Fourier transform of $s_i(t)$. Then the

cross-interference power which signal j causes in a filter matched to signal i averaged over an interval $T_A > 2T$ is

$$P_{ij} = \frac{1}{T_A} \int_{-\infty}^{\infty} |S_i(f)|^2 |S_j(f)|^2 \, df \qquad (1)$$

To prove this assertion let

$$R_{ij}(\tau) = \int_{-\infty}^{\infty} s_i(t) \, s_j(\tau - t) \, dt \quad .$$

This is the cross-correlation of $s_i(t)$ and $s_j(t)$, and represents the waveform at the output of a filter matched to s_i when s_j is the input. Notice that since $s_i(t)$ has duration T, $R_{ij}(\tau)$ will have duration 2T. Suppose that the matched filter output is sampled at $t = 0$, and suppose that the signal $s_j(t)$ arrives at some random time τ, where τ is uniformly distributed over the interval $\left(-T_A/2, T_A/2\right)$. Then the average power of the sampled output will be

$$P_{ij} = E\left|R_{ij}^2(\tau)\right|$$

$$= \frac{1}{T_A} \int_{-T_A/2}^{T_A/2} R_{ij}^2(\tau) \, d\tau$$

If $T_A > 2T$, then since $R_{ij}(\tau) = 0$ outside the interval $-T \le \tau \le T$, we can replace the limits on the above integral by $\pm \infty$, so that

$$P_{ij} = \frac{1}{T_A} \int_{-\infty}^{\infty} R_{ij}^2(\tau) \, d\tau$$

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Applying Parseval's theorem to this integral results in an expression for P_{ij} in terms of the Fourier transform of $R_{ij}(\tau)$:

$$P_{ij} = \frac{1}{T_A} \int_{-\infty}^{\infty} \left| F\left| R_{ij}(\tau) \right| \right|^2 df \quad .$$

But the convolution-multiplication properties of Fourier transforms are such that

$$F\left| R_{ij}(\tau) \right| = S_i(f)\, S_j^*(f)$$

so that

$$P_{ij} = \frac{1}{T_A} \int_{-\infty}^{\infty} |S_i(f)|^2 |S_j(f)|^2 df \quad .$$

In most communication systems each transmitted message will consist of a sequence of many symbols, and the receiver must make a decision at the end of each symbol. Thus, the averaging time of interest is just two symbols long, and each symbol lasts the duration of one signaling waveform, so that $T_A = 2T$. Only these cases are considered in the rest of this note.

Achieving Minimum Cross Interference Power

Equation (1) suggests that one approach to design multiple-access signals is to select mutually-orthogonal spectral-density functions. This is difficult to do in the time-domain, but easily accomplished in the frequency domain, where it leads to frequency-division multiple-access systems. Notice that since $|S_i(f)|^2$ can never be negative, the only way to obtain mutually orthogonal signals in the absence of time-synchronization is to assure that the spectral density functions do not overlap. This, of course, is the definition of a frequency-division multiple-access system. It is interesting to note, that no

other multiple access technique can possibly achieve zero average cross interference without time synchronization. In this sense, frequency-division multiple access is optimum. Of course, signals which are time-limited cannot be bandlimited, so that perfect spectral orthogonality cannot be achieved. Nevertheless, it is quite practical to construct a frequency division system which will result in cross interference which is at or below the noise level.

Minimum Cross-Interference Power for Spread Spectrum Signals

Spread spectrum signals generated for multiple-access systems usually are drawn from a set of signals with identical spectral density functions $|S_i(f)|^2$. This is because of the economic need to make all signal-generators alike, and to obtain different signals by using different key-streams. The spectral-density function is usually determined by the transmitting filter used to limit the occupancy bandwidth of the signals. Signal differences are usually contained in the detailed phase structure of the signal, which does not affect the spectral-density function. Examples of such signal sets include filtered and non-filtered PSK and MSK binary chip sequences where the sequences are all at the same chip rate. Complementary binary sequences and maximal-length sequences with a fixed chip rate both result in signal sets with identical spectral density functions.

In these cases, which are of particular interest to the design of spread-spectrum multiple-access systems, one can show that the maximum cross-interference rejection which can be obtained is 2TW, where T is the bit duration and W is the occupied bandwidth. More specifically, the following theorem applies.

Theorem 1:

Let F(x) be a continuous, real function on the real line everywhere ≥ 0 such that

$$\int_{-\infty}^{\infty} F(x)dx = 1$$

and

$$\int_{-\frac{W}{2}}^{\frac{W}{2}} F(x)dx = \eta_W \quad , \tag{2}$$

where

$$0 < \eta_W < 1$$

Then

$$\int_{-\infty}^{\infty} F^2(x)dx \geq \frac{\eta_W^2}{W}$$

Proof:

To prove this theorem one need only apply the calculus of variations. Let

$$I = \int_{-\frac{W}{2}}^{\frac{W}{2}} F^2(x)dx$$

and find the F(x) which minimizes I subject to the constraint (2). First define a new function

$$G(F, F', x) = F^2 + \lambda F$$

where λ is a Lagrange multiplier. Then solve the Euler-Lagrange equation:

$$\frac{\partial G}{\partial F} = \frac{d}{dx}\left(\frac{\partial G}{\partial F'}\right)$$

where $F' = \frac{dF}{dx}$. This results in the equation

$$2F + \lambda = 0$$

or $F = \frac{\lambda}{2}$. Substituting this into the constraint (2) results in

$$\int_{-\frac{W}{2}}^{\frac{W}{2}} \frac{\lambda}{2}\, dx = \eta_w \quad , \text{ or}$$

$$\lambda = \frac{2\,\eta_w}{W} \quad .$$

Finally, solve for I_{min},

$$I_{min} = \int_{-\frac{W}{2}}^{\frac{W}{2}} \frac{\eta_w^2}{W^2}\, dx = \frac{\eta_w^2}{W} \quad .$$

Now since F(x) is real and positive, then $F^2(x)$ will be real and positive and therefore

$$\int_{-\infty}^{\infty} F^2(x)dx \geq \int_{-\frac{W}{2}}^{\frac{W}{2}} F^2(x)dx \geq \frac{\eta_w^2}{W}$$

which proves the theorem.

Now let us apply this theorem to equation (1) and find the minimum cross interference power for signals with identical spectral density functions. Let the spectral density function of a set of signals be

$P_k F(f)$, where P_k represents the power in the received signal $S_k(t)$. Then equation (1) becomes

$$P_{ij} = \frac{1}{2T} P_i P_j \int_{-\infty}^{\infty} F^2(f)\, df$$

If we let W be the bandwidth of the signals which contains η_w of the power, then the theorem shows that

$$P_{ij} \geq \frac{P_i P_j \eta_w^2}{2TW}$$

Now, since P_i is the power in the impulse response of a filter matched to signal i, it should be normalized to 1, and we see that the cross interference power is reduced from the interfering power P_j by a factor of at best $\eta_w^2 / 2TW$. Notice that for bandlimited signals in which essentially all the power is concentrated in an occupied band of width W, $\eta_w^2 \to 1$.

Application to Some Signal Sets

Uniform Spectral Density

Consider the set of signals with uniform spectral density over the band W. Theorem 1 shows that this is an optimal signal set. In this case the signal spectrum will be

$$|S_j(f)|^2 = \frac{P_j}{W} \qquad -\frac{W}{2} \leq f \leq \frac{W}{2}$$

$$= 0 \qquad \text{elsewhere}$$

where P_j = average power during signaling interval. The matched filter spectral response will be normalized so that

$$|S_i(f)|^2 = \frac{1}{W} \qquad -\frac{W}{2} \le f \le \frac{W}{2}$$

$$= 0 \qquad \text{elsewhere}$$

Substituting these expressions into equation (1) with $T_A = 2T$,

$$P_{ij} = \frac{P_j}{2TW} .$$

Notice that this equation holds true of <u>any set of common-channel signals</u> which have uniform spectral density and equal symbol duration.

<u>Gaussian Spectral Density</u>

A set of signals with Gaussian spectral density with rms bandwidth B will have spectral density

$$|S_j(f)|^2 = \frac{P_j}{\sqrt{2\pi}\, B} \exp\left(-f^2/2B^2\right)$$

and normalized filter response

$$|S_i(f)|^2 = \frac{1}{\sqrt{2\pi}\, B} \exp\left(-f^2/2B^2\right) .$$

The cross interference power is

$$P_{ij} = \frac{P_j}{2\pi B\ (2TB)} \int_{-\infty}^{\infty} \exp\left(-2f^2/2B^2\right) df$$

$$= \frac{P_j}{2T\ (2\sqrt{\pi}\, B)}$$

The quantity $2\sqrt{\pi}\,B$ is the 92 percent bandwidth of the signal, and roughly comparable to the bandwidth W for uniform spectral density signals.

Sinc^2 Spectral Density

A set of signals with sinc^2 spectral density with null-to-null bandwidth B will have spectral density

$$|S_j(f)|^2 = \frac{P_j}{\pi B} \frac{\sin^2(2\pi f/B)}{(2\pi f/B)^2}$$

and normalized filter response

$$|S_i(f)|^2 = \frac{1}{\pi B} \frac{\sin^2(2\pi f/B)}{(2\pi f/B)}$$

The cross interference power will be

$$P_{ij} = \frac{P_j}{2T\left(\frac{3\pi B}{4}\right)}$$

The null-to-null bandwidth B is not the 90 percent bandwidth for this set of signals. On the other hand, $\frac{3\pi B}{4}$ contains more than 90 percent of the energy, and is roughly comparable to W for the uniform spectral density signals.

N-Channel Signals

Suppose that each of the M signals occupies N channels. Such signals might be generated, for example, by transmitting the two parts of a set of complementary-sequence signals over two isolated frequency bands. In this case $|S_i(f)|^2$ and $|S_j(f)|^2$ will each have N parts; e.g.,

$$|S_j(f)|^2 = P_j\left(F_1(f+f_1) + F_2(f+f_2) + \cdots + F_N(f+f_N)\right)$$

Clearly, so long as all signals have the same spectral density, Theorem 1 applies. However, W must be interpreted as the occupied bandwidth (the sum of the channel bandwidths), ignoring the unoccupied bands between channels, to obtain a tighter bound. Thus if W_i is the bandwidth of channel i, then

$$P_{ij} \geq \frac{P_j}{2T\sum_i W_i}$$

Conclusion

It has been shown that the cross-interference between binary asynchronous multiple-access signals is the integral of the product of the spectral densities divided by twice the bit duration. Frequency-division multiple access schemes can achieve near zero cross-interference power. No other asynchronous schemes can achieve this minimum. Spread spectrum schemes using signals with identical spectral density functions can achieve a minimum cross interference power which is the power in the interfering signals divided by 2TW where T is the bit duration and W is the occupied bandwidth. Most reasonable spread-spectrum schemes, including filtered and unfiltered PSK, MSK, and complementary sequence signals will achieve this minimum.